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Superdiffusive Behaviour of a Passive Ornstein–Uhlenbeck Tracer in a Turbulent Shear Flow

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In this paper, we are interested in the study of the diffusion of a passive particle with positive mass by a divergence free velocity field. We consider here the very simple turbulent shear flow case, in which we will prove the superdiffusive behaviour of the motion for large enough values of the energy spectrum of the velocity field. For small values, the proof of the diffusive behaviour of the model is also new, and it is shown that this diffusion is strictly greater than the one obtained with a non-massive particle. One interesting point to insist on is that we are able to obtain explicit hydrodynamic equations without even having the stationary measure of the studied processes.

KEY WORDS: Superdiffusion; turbulent; shearflow; Ornstein–Uhlenbeck; hydrodynamic; tagged particle.

1. INTRODUCTION

1.1. Objects and Notations

Let us consider the following simple model: the so-called Ornstein–Uhlenbeck process in a random velocity field depending only on the X direction and acting only on the Y direction. This process is solution of the following system of stochastic differential equations

$$\begin{cases} dX_t = u_t dt \\ \sigma du_t = -u_t dt + \sqrt{\mu}\ d\beta_t \\ dY_t = v_t dt \\ \tau dv_t = -v_t dt - \gamma_\delta(X_t)dt + \sqrt{\nu}\ d\beta'_t \end{cases} \tag{1}$$

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with initial conditions

$$X_0 = Y_0 = 0.$$

u_0 and v_0 are fixed and deterministic, β and β' are standard independant brownian motions and μ and ν are positive real numbers representing the bare diffusivity, also called molecular diffusivity. The parameters σ, τ, μ, ν, are strictly positive but in a small discussion at the begining of this article, we consider briefly the other cases. Note that it is interesting here to consider the $\mu \neq \nu$ case, which allows us to make some qualitative remarks. γ_δ is the gaussian centered field

$$\gamma_\delta(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{ikx} |k|^{\frac{1-\varepsilon}{2}} \psi_0^{1/2}\left(\frac{|k|}{\delta}\right) \psi_\infty^{1/2}(|k|) dw_k \tag{2}$$

with covariance

$$\left\langle \gamma_\delta(x)\gamma_\delta(x')\right\rangle = \frac{1}{2\pi} \int_{\mathbb{R}} \mathrm{e}^{ik(x-x')} |k|^{1-\varepsilon} \psi_0\left(\frac{|k|}{\delta}\right) \psi_\infty(|k|) dk \tag{3}$$

where δ is a positive real number. w_k denotes a gaussian complex white noise verifying

$$\langle dw_k d\bar{w}_{k'}\rangle = \delta(k-k')dk.$$

ψ_0 and ψ_∞ are respectively infrared and ultraviolet cut-offs, defined above $[0,\infty)$. They verify $0 \leqslant \psi_i \leqslant 1$, and there exists k_0, k_1 real numbers such that $0 < k_0 < k_1$ and $\psi_0(k) = 0$ if $k < k_0$, $\psi_1(k) = 0$ if $k > k_1$. At last, ψ_0 goes to 1 at ∞, and ψ_∞ is continuous at 0 and $\psi_\infty(0) = 1$.

In the rest of the article, we will use the following notation

$$\hat{F}_{\delta,\varepsilon}(k) = |k|^{1-\varepsilon} \psi_0\left(\frac{|k|}{\delta}\right) \psi_\infty(|k|). \tag{4}$$

$\hat{F}_{\delta,\varepsilon}(k)$ is called the *energy spectrum* of the field.

$\langle . \rangle$ denotes the expectation with respect to the distribution of γ_δ, $\mathbb{E}_{u_0,v_0}$ denotes the expectation with respect to the joint distribution of β and β' with u_t and v_t starting at u_0 and v_0. All of the previous random objects are chosen independent.

Let us also introduce the infinitesimal generator L_δ of the Markov process (X, Y, u, v)

$$L_\delta = \frac{1}{2}\left(\frac{\mu}{\sigma^2}\partial_u^2 + \frac{\nu}{\tau^2}\partial_v^2\right) - \left(\frac{u}{\sigma}\partial_u + \frac{v}{\tau}\partial_v\right) + u\partial_x + v\partial_y - \frac{\gamma_\delta(x)}{\tau}\partial_v$$

2. MAIN RESULTS

2.1. Goals

Let ρ be a positive real number and let T_δ be the solution of

$$\partial_t T_\delta(x,y,u,v,t) = L_\delta T_\delta(x,y,u,v,t)$$

with initial condition

$$T_\delta(x,y,u,v,0) = T_0(\delta x, \delta y)$$

For any fixed ε in $\mathbb{R}$, we are looking for the existence of a suitable scaling factor $\rho(\delta,\varepsilon)$ so that the following limit exists

$$\bar{T}(x,y,u,v,t) = \lim_{\delta\to 0}\left\langle T_\delta\left(\frac{x}{\delta}, \frac{y}{\delta}, \frac{\rho^2 u}{\delta}, \frac{\rho^2 v}{\delta}, \frac{t}{\rho^2}\right)\right\rangle \tag{5}$$

and we also want to compute the equation verified by $\bar{T}$ (in fact we will need to restrain ourselves to $\varepsilon \leqslant 4$).

2.2. Results

In this article, we will show the three following theorems

Theorem 1. When $\varepsilon < 0$, under a diffusive space-time scaling

$$\rho = \delta$$

the function $\bar{T}$ defined by (5) does not depend on u and v, and verifies the following simple diffusion equation

$$\partial_t \bar{T} = \left[\frac{\mu}{2}\partial_x^2 + \frac{\nu + D(\varepsilon)}{2}\partial_y^2\right]\bar{T}$$

with

$$D(\varepsilon) = \frac{1}{\pi}\int_{\mathbb{R}} \mathrm{k}|k|^{1-\varepsilon}\psi_\infty(|k|)\int_0^{+\infty} \mathrm{e}^{-\frac{k^2\mu\sigma}{2}(s/\sigma - 1 + \mathrm{e}^{-s/\sigma})} ds. \tag{6}$$

Theorem 2. When $2<\varepsilon<4$, the suitable value for ρ is

$$\rho=\delta^{1-\varepsilon/4}$$

and $\bar{T}$ does not depend on u and v and verifies the equation

$$\partial_t \bar{T}=\frac{1}{2} t D(\varepsilon) \partial_y^2 \bar{T}$$

with

$$D(\varepsilon)=\frac{1}{\pi} \int_{\mathbb{R}}|k|^{1-\varepsilon} \psi_0(|k|) d k.$$

Theorem 3. When $0<\varepsilon<2$, the suitable value for ρ is

$$\rho=\delta^{\frac{1}{1+\varepsilon/2}}.$$

For fixed $\alpha>0$, we introduce the solution T_α of

$$\partial_t T_\alpha=\left(1+\frac{\varepsilon}{2}\right) \frac{t^{\varepsilon/2} \mu^{\varepsilon/2-1}}{\sqrt{8\pi}} \alpha \partial_y^2 T_\alpha.$$

Then, one can show that $\bar{T}$ does not depend on u and v and verifies the equation

$$\bar{T}(x, y, t)=\int_0^{\infty} T_\alpha(x, y, t) \nu_{\varepsilon}(d \alpha)$$

for a suitable measure ν_ε which will be defined in the proof.

2.3. Remarks and Commentaries

The superdiffusive behaviour of a non-massive tracer in a turbulent divergence-free flow has been studied many times, see for instance refs. 2–4 and 7 in the case of a discrete energy spectrum. For a study in a non-stratified case, see ref. 6. In the diffusive cases, one can prove a very general result, see ref. 5.

2.3.1. Comparison with the Brownian Motion Case

Let us recall the following results, which are all contained in the article[1] by M. Avellaneda and A. Majda, and which inspired this article.

Proposition 2.1. Using the same notations as previously, let us consider the following system

$$\begin{cases} \tilde{X}_t = \sqrt{\mu}\beta_t \\ d\tilde{Y}_t = -\gamma_\delta(\tilde{X}_t)dt + \sqrt{\nu}d\beta'_t \end{cases}$$

and let us introduce $\tilde{L}_\delta$ the generator of $(\tilde{X}, \tilde{Y})$

$$\tilde{L}_\delta = \frac{1}{2}\left(\mu\partial_x^2 + \nu\partial_y^2\right) - \gamma_\delta(x)\partial_y$$

and $\tilde{T}_\delta$ the solution of

$$\partial_t \tilde{T}_\delta(x, y, t) = \tilde{L}_\delta \tilde{T}_\delta(x, y, t)$$

with initial condition

$$\tilde{T}_\delta(x, y, 0) = \tilde{T}_0(\delta x, \delta y).$$

We are also looking for a suitable scaling factor $\rho(\delta, \varepsilon)$ such that

$$\bar{\tilde{T}}(x, y, t) = \lim_{\delta \to 0} \left\langle \tilde{T}_\delta\left(\frac{x}{\delta}, \frac{y}{\delta}, \frac{t}{\rho^2}\right)\right\rangle$$

exists. Then

- For $\varepsilon < 0$, $\rho = \delta$, the limit exists and verifies

$$\partial_t \bar{\tilde{T}} = \left[\frac{\mu}{2}\partial_x^2 + \frac{\nu + \tilde{D}(\varepsilon)}{2}\partial_y^2\right]\bar{\tilde{T}}$$

with

$$\tilde{D}(\varepsilon) = \frac{2}{\mu\pi}\int_{\mathbb{R}} dk|k|^{-1-\varepsilon}\psi_\infty(|k|).$$

- In all the other cases, i.e. $0 \leqslant \varepsilon < 4$, the suitable values for ρ are the same in the two models, and the limit $\bar{\tilde{T}}$ verfies exactly

$$\bar{\tilde{T}} = \bar{T}.$$

It is interesting to notice that the two models have the same behaviour under space-time rescaling, with the same scaling factors. In the diffusive case, we have a quantitative difference between the two limits, as it is easy to show that for all $\sigma > 0$

$$\tilde{D}(\varepsilon) < D(\varepsilon), \tag{7}$$

while this difference vanishes in the superdiffusive cases. One can explain (7) because of the superior inertia of the Ornstein–Uhlenbeck process which makes it less sensitive to a turbulent environment. The reader will also notice that simply making $\sigma = 0$ in the superdiffusive cases do not change the proof, and the results for the massless particle are included in those for a massive particle in these cases.

2.3.2. Dependance on the Various Parameters

The parameter τ. As one can notice, none of the diffusion equations depend on the parameter τ. Referring to the previous remark, this means that an intermediate model, where X would be an Ornstein–Uhlenbeck process and Y a simple brownian motion in a random velocity field γ would not be treated differently as the model proposed in this article. Making $\tau = 0$ would only imply obvious changes.

The parameters μ *and* ν. In the superdiffusive cases $\varepsilon \geqslant 0$, there is no explicit dependance on the parameter ν, while in the diffusive case, this dependance is not significant, while making $\nu = 0$ does not change the model. In fact, it is not necessary to introduce a pertubation term for the Y coordinate to obtain a valid model. On the contrary, making $\mu = 0$ would force the X coordinate to stay in a bounded layer and then, one would not observe any diffusion under space-time rescaling.

2.3.3. The Time Dependant Case

The interested reader will easily treat the case where γ_δ is not stationnary, referring to this article and the one of M. Avellaneda and A. Majda.[1]

2.3.4. The Other Cases

The interested reader will show the following assertions

Proposition 2.2. For $\varepsilon = 0$, the suitable scaling factor is

$$\rho = \delta\sqrt{\log(-\delta)}$$

and the effective diffusion equation for $\bar{T}$ is

$$\partial_t \bar{T} = \frac{\mu}{\pi}\partial_y^2 \bar{T}.$$

For $\varepsilon = 2$, the suitable scaling factor is

$$\rho = \delta^{1/2}\log(-\delta)^{1/4}$$

and the effective diffusion equation for $\bar{T}$ is

$$\partial_t \bar{T} = \frac{t}{4\pi}\partial_y^2 \bar{T}.$$

These results are the same for the Brownian motion case[1].

3. PRELIMINARY CALCULUS

Before we begin to prove the theorems, let us make some easy calculus which will be useful in all of the cases.
We have

$$\left\langle T_\delta\left(\frac{x}{\delta}, \frac{y}{\delta}, \frac{\rho^2 u}{\delta}, \frac{\rho^2 v}{\delta}, \frac{t}{\rho^2}\right)\right\rangle = \left\langle \mathbb{E}_{\rho^2\delta^{-1}u, \rho^2\delta^{-1}v}[T_0(\delta X_{\rho^{-2}t} + x, \delta Y_{\rho^{-2}t} + y)]\right\rangle.$$

In order to simplify the notations, we will write in all the following calculus

$$\mathbb{E}_{\rho^2\delta^{-1}u, \rho^2\delta^{-1}v} = \mathbb{E}.$$

Introducing the Fourier transform $\hat{T}_0$ of T_0, we obtain

$$\begin{aligned} &\left\langle \mathbb{E}\left[T_0(\delta X_{\rho^{-2}t} + x, \delta Y_{\rho^{-2}t} + y)\right]\right\rangle \\ &\quad = \frac{1}{2\pi}\iint_{\mathbb{R}^2} \hat{T}_0(\eta, \xi) e^{i\eta x + i\xi y}\left\langle \mathbb{E}\left[e^{i\eta\delta X_{\rho^{-2}t} + i\xi\delta Y_{\rho^{-2}t}}\right]\right\rangle d\eta d\xi. \qquad (8) \end{aligned}$$

On the other hand, one can verify that

$$d(\mathrm{e}^{t/\sigma}u_t)=\mathrm{e}^{t/\sigma}\frac{\sqrt{\mu}}{\sigma}d\beta_t$$

so that

$$u_t=\delta u\mathrm{e}^{-t/\sigma}+\frac{\sqrt{\mu}}{\sigma}\int_0^t \mathrm{e}^{(s-t)/\sigma}d\beta_s. \tag{9}$$

The same calculus shows that we have

$$v_t=\delta v\mathrm{e}^{-t/\tau}-\frac{1}{\tau}\int_0^t \mathrm{e}^{(s-t)/\tau}\gamma_\delta(X_s+x/\delta)ds+\frac{\sqrt{\nu}}{\tau}\int_0^t \mathrm{e}^{(s-t)/\tau}d\beta'_s$$

and (1) implies

$$dY_t=-\tau dv_t-\gamma_\delta(X_t)dt+\sqrt{\nu}d\beta'_t$$

so that

$$\begin{aligned}Y_t &= \delta\tau v-\tau v_t-\int_0^t \gamma_\delta(X_s+x/\delta)ds+\sqrt{\nu}\beta'_t\\ &=(1-\mathrm{e}^{-t/\tau})\delta\tau v-\int_0^t (1-\mathrm{e}^{(s-t)/\tau})\gamma_\delta(X_s+x/\delta)ds\\ &\quad+\sqrt{\nu}\int_0^t(1-\mathrm{e}^{(s-t)/\tau})d\beta'_s\end{aligned} \tag{10}$$

Now, going back to (8), we obtain finally the following formula

$$\begin{aligned}&\left\langle\mathbb{E}\left[T_0(\delta X_{\rho^{-2}t}+x,\delta Y_{\rho^{-2}t}+y)\right]\right\rangle\\ &=\tfrac{1}{2\pi}\int\int_{\mathbb{R}^2}d\eta d\xi\,\hat{T}_0(\eta,\xi)\mathrm{e}^{i\eta x+i\xi y}\langle\mathbb{E}\big[\mathrm{e}^{i\eta\delta X_{\rho^{-2}t}}\\ &\qquad\times\mathrm{e}^{i\xi\delta\sqrt{\nu}\int_0^{\rho^{-2}t}(1-\mathrm{e}^{(s-\rho^{-2}t)/\tau})d\beta'_s}\\ &\qquad\times\mathrm{e}^{i\xi\tau\rho^2(1-\mathrm{e}^{-\rho^{-2}t/\tau})v}\\ &\qquad\times\mathrm{e}^{-i\xi\delta\int_0^{\rho^{-2}t}(1-\mathrm{e}^{(s-\rho^{-2}t)/\tau})\gamma_\delta(X_s+\delta^{-1}x)ds}\big]\rangle.\end{aligned} \tag{11}$$

4. THE DIFFUSIVE CASE: ε<0

In this section, we will suppose that ε is a fixed negative real number. In this case, one can see that the integral

$$\int_{\mathbb{R}} \frac{dk}{k^2} |k|^{1-\varepsilon} \psi_\infty(|k|)$$

is convergent. Then, reffering to ref. 1, we expect that a simple diffusive scaling $\rho = \delta$ will be suitable and that $\bar{T}$ will verify a simple diffusion equation of the form

$$\partial_t \bar{T} = \left[\frac{\mu}{2} \partial_x^2 + \frac{\nu + D(\varepsilon)}{2} \partial_y^2 \right] \bar{T}$$

where $D(\varepsilon)$ depends only on μ, σ, ε and ψ_∞. We will prove this result and then try to compute the value of D using an other means. It is then quite easy to show that this diffusion coefficient is strictly greater than its equivalent in ref. 1.

4.1. The Trivial Terms

From now on, let us consider that $\rho = \delta$. Going back to (11), we see that we have several terms to study, but it is quite easy to show that

$$\delta \int_0^{\delta^{-2}t} \left(1 - \mathrm{e}^{(s-\delta^{-2}t)/\tau} \right) d\beta'_s \to \beta'_t$$

in distribution when δ goes to 0. It is also clear that

$$\delta^2 (1 - \mathrm{e}^{-\delta^{-2}t/\tau}) v \to 0$$

pointwise in v when δ goes to 0. It is not more difficult to show that

$$\delta \mathrm{e}^{-\delta^{-2}t/\tau} \int_0^{\delta^{-2}t} \mathrm{e}^{s/\tau} \gamma_\delta (X_s + \delta^{-1} x) ds \to 0$$

in L^2 when δ goes to 0, using the following calculus

$$\left\langle \mathbb{E}\left[\left|\delta \mathrm{e}^{-\delta^{-2}t/\tau}\int_0^{\delta^{-2}t}\mathrm{e}^{s/\tau}\gamma_\delta(X_s+\delta^{-1}x)ds\right|^2\right]\right\rangle$$
$$=\frac{\delta^2\mathrm{e}^{-2\delta^{-2}t/\tau}}{2\pi}\int_0^{\delta^{-2}t}\int_0^{\delta^{-2}t}dsds'\mathrm{e}^{\frac{s+s'}{\tau}}\int_{\mathbb{R}}\mathbb{E}\left[\mathrm{e}^{ik(X_s-X_{s'})}\right]|k|^{1-\varepsilon}\psi_0\left(\frac{|k|}{\delta}\right)\psi_\infty(|k|)dk$$
$$\leqslant\frac{\delta^2}{2\pi}\mathrm{e}^{-2\delta^{-2}t/\tau}\left(\int_0^{\delta^{-2}t}\mathrm{e}^{s/\tau}ds\right)^2\int_{\mathbb{R}}|k|^{1-\varepsilon}\psi_\infty(|k|)dk$$

Eventually, using independance of the various random objects, we only have to show that the following quantity

$$\left\langle \mathbb{E}\left[\mathrm{e}^{i\delta(\eta X_{\delta^{-2}t}-\xi\int_0^{\delta^{-2}t}\gamma_\delta(X_s+x/\delta)ds}\right]\right\rangle$$

has a limit when δ goes to 0, and we must compute this limit.

4.2. Introduction of a Useful Martingale

Let us introduce the solution $U_\delta(x,u)$ of the following partial differential equation

$$\left(\frac{\mu}{2\sigma^2}\partial_u^2-\frac{1}{\sigma}u\partial_u+u\partial_x\right)U_\delta(x,u)=\gamma_\delta(x) \tag{12}$$

Considering the definition (2) of γ_δ, we define f_k as

$$U_\delta(x,u)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\mathrm{e}^{ikx}f_k(u)|k|^{\frac{1-\varepsilon}{2}}\psi_0^{1/2}(|k|/\delta)\psi_\infty^{1/2}(|k|)dw_k. \tag{13}$$

Then f_k verifies

$$\mu f_k''-2\sigma uf_k'+2ik\sigma^2uf_k=2\sigma^2. \tag{14}$$

We choose for f_k the solution of (14) with initial conditions

$$\begin{cases} f_k(0)=0\\ f_k'(0)=0\end{cases}$$

We have the following lemma

Lemma 4.1.

$$\begin{cases} \displaystyle\int_{-\infty}^{\infty} |f_k'(u)|^2 \mathrm{e}^{-\sigma u^2/\mu} \frac{du}{\sqrt{\pi\mu\sigma^{-1}}} \leqslant \frac{2\sigma^3}{\mu} + \frac{8\sigma^2}{\mu^2 k^2} \\ \displaystyle\int_{-\infty}^{\infty} |f_k(u)|^2 \mathrm{e}^{-\sigma u^2/\mu} \frac{du}{\sqrt{\pi\mu\sigma^{-1}}} \leqslant 2\sigma^2 + \frac{12\sigma}{\mu k^2} + \frac{16}{\mu^2 k^4} \end{cases} \tag{15}$$

for all $k>0$.

$\diamondsuit$ Let us introduce the following scalar product for all suitable functions ϕ, ψ

$$[[\phi,\psi]] = \int_{-\infty}^{\infty} \phi(u)\psi(u)\mathrm{e}^{-\sigma u^2/\mu} \frac{du}{\sqrt{\pi\mu\sigma^{-1}}}.$$

One can see that the differential operator

$$S\phi(u) = \mu\phi''(u) - 2\sigma u\phi'(u) \tag{16}$$

is symmetric for [[.]] and that

$$[[S\phi,\psi]] = -\mu\,[[\phi',\psi']]\,.$$

Let us also notice that for any ϕ

$$[[u\phi]] = \frac{\mu}{2\sigma}\,[[\phi']]\,.$$

At last, it is easy to show that the real part of f_k, $\mathcal{R}(f_k)$, is even, and that its imaginary part $\mathcal{I}(f_k)$ is odd.

Then we have, applying [[.]] to (14)

$$[[f_k']] = -\frac{2\sigma i}{\mu k} \tag{17}$$

and multiplying (14) by $\bar{f}_k$ and applying [[.]]

$$-\mu\left[\left[|f_k'|^2\right]\right] = 2\sigma^2\,[[\bar{f}_k]] = 2\sigma^2\,[[f_k]]\,.$$

Now, let us multiply (14) by u and apply once again [[.]]

$$-\left[\left[f_k'\right]\right]+ik\sigma\left[\left[uf_k'\right]\right]+ik\sigma\,[[f_k]]=0$$

so that

$$[[f_k]]=-\frac{2}{\mu k^2}-\left[\left[uf_k'\right]\right]$$

and then, using Shwarz inequality

$$\frac{\mu}{2}\left[\left[|f_k'|^2\right]\right]=\frac{2\sigma^2}{\mu k^2}+\sigma^2\left[\left[uf_k'\right]\right]\leqslant\frac{2\sigma^2}{\mu k^2}+\sqrt{\frac{\mu\sigma^3}{2}}\left[\left[|f_k'|^2\right]\right]^{1/2}$$

which gives easily

$$\left[\left[|f_k'|^2\right]\right]\leqslant\frac{2\sigma^3}{\mu}+\frac{8\sigma^2}{\mu^2k^2}.$$

Now, using Poincaré's inequality, considering that the spectral gap of the operator S defined by (16) is equal to 2σ, we have

$$\left[\left[\left|f_k+\frac{\mu}{2\sigma^2}\left[\left[|f_k'|^2\right]\right]\right|^2\right]\right]\leqslant\frac{\mu}{2\sigma}\left[\left[|f_k'|^2\right]\right]$$

so

$$\left[\left[|f_k|^2\right]\right]+\frac{\mu}{\sigma^2}\,[[\mathcal{R}(f_k)]]\left[\left[|f_k'|^2\right]\right]+\frac{\mu^2}{4\sigma^4}\left[\left[|f_k'|^2\right]\right]^2\leqslant\frac{\mu}{2\sigma}\left[\left[|f_k'|^2\right]\right]$$

and then

$$\left[\left[|f_k|^2\right]\right]\leqslant\frac{\mu}{2\sigma}\left[\left[|f_k'|^2\right]\right]+\frac{\mu^2}{4\sigma^4}\left[\left[|f_k'|^2\right]\right]^2\leqslant 2+\frac{12\sigma}{\mu k^2}+\frac{16}{\mu^2k^4}.\ \diamond$$

This lemma will allow us to write all the following integrals and expectations. Let us remark that (17) also shows that

$$\left[\left[|f_k'|^2\right]\right]\geqslant\frac{4\sigma^2}{\mu^2k^2}$$

and consequently, one will easily verify that the following results cannot be extended to the $\varepsilon \geqslant 0$ cases.

Using Ito's formula, we obtain

$$\delta\left(\eta X_{\delta^{-2}t} - \xi\int_0^{\delta^{-2}t}\gamma_\delta(X_s + x/\delta)ds\right) = -\xi\delta U_\delta(X_{\delta^{-2}t} + x/\delta, u_{\delta^{-2}t}) + \xi\delta U_\delta(x/\delta, \delta u) + \tilde{M}_\delta(t) + \eta\delta X_{\delta^{-2}t} \tag{18}$$

where $\tilde{M}_\delta$ is the martingale

$$\tilde{M}_\delta(t) = \delta\frac{\sigma}{\sqrt{\mu}}\int_0^{\delta^{-2}t}\xi\,\partial_u U_\delta(X_s + x/\delta, u_s)d\beta_s$$

As we have

$$X_t = \sqrt{\mu}\beta_t - \sigma u_t + \delta\sigma u$$

We introduce the martingale

$$M_\delta(t) = \delta\int_0^{\delta^{-2}t}\left[\sqrt{\mu}\eta + \xi\frac{\sigma}{\sqrt{\mu}}\partial_u U_\delta(X_s + x/\delta, u_s)\right]d\beta_s$$

with quadratic variation

$$Q_\delta(t) = \delta^2\int_0^{\delta^{-2}t}\left|\sqrt{\mu}\eta + \xi\frac{\sigma}{\sqrt{\mu}}\partial_u U_\delta(X_s + x/\delta, u_s)\right|^2 ds$$

and (18) becomes

$$\delta\left(\eta X_{\delta^{-2}t} - \xi\int_0^{\delta^{-2}t}\gamma_\delta(X_s + x/\delta)ds\right) = -\xi\delta U_\delta(X_{\delta^{-2}t} + x/\delta, u_{\delta^{-2}t}) + \xi\delta U_\delta(x/\delta, \delta u) + M_\delta(t) + \eta\sigma\delta u_{\delta^{-2}t} + \delta^2\sigma u \tag{19}$$

Q_δ verifies

$$\begin{aligned}
&\mathbb{E}[\langle Q_\delta(t)\rangle] \\
&= \delta^2 \int_0^{\delta^{-2}t} ds \left[\mu\eta^2 + \frac{\xi^2\sigma^2}{2\pi\mu} \int_{\mathbb{R}} dk|k|^{1-\varepsilon} \psi_0(|k|/\delta)\psi_\infty(|k|)\mathbb{E}[|f_k'(u_s)|^2] \right] \qquad (20) \\
&= \eta^2\mu t + \frac{\xi^2\sigma^2}{2\pi\mu} \int_{\mathbb{R}} dk|k|^{1-\varepsilon} \psi_0(|k|/\delta)\psi_\infty(|k|) \int_0^t \mathbb{E}[|f_k'(u_{\delta^{-2}s})|^2] ds.
\end{aligned}$$

Let us notice that, as u_t is a gaussian process with variance

$$\frac{\mu}{2\sigma}(1 - \mathrm{e}^{-2t/\sigma})$$

(see (9)), the existence of all those integrals are ensured by (15).

4.3. Asymptotic behaviour of u_t

The following lemma holds

Lemma 4.2. When t goes to infinity, u_t converges almost surely to a gaussian centered random variable u_∞ with variance $\mu/2\sigma$. Moreover, for all $t \geqslant 0$, the variance of u_t is less than $\mu/2\sigma$.

◇ Referring to (9), we have

$$u_t = \delta u \mathrm{e}^{-t/\sigma} + \frac{\sqrt{\mu}}{\sigma} \int_0^t \mathrm{e}^{(s-t)/\sigma} d\beta_s.$$

Let $\mathcal{F}_t$ be the filtration generated by u_t. We have for all $0 \leqslant t \leqslant t'$

$$\mathbb{E}[u_{t'}|\mathcal{F}_t] = \delta u \mathrm{e}^{-t'/\sigma} + \mathrm{e}^{-t'/\sigma} \frac{\sqrt{\mu}}{\sigma} \int_0^t \mathrm{e}^{s/\sigma} d\beta_s \leqslant u_t$$

so that u_t is a supermartingale. Moreover, one can show that

$$\mathbb{E}[(u_t - \delta u)^-] = \int_{-\infty}^0 -x \frac{\mathrm{e}^{-\frac{\sigma x^2}{\mu(1-\mathrm{e}^{-2t/\sigma})}}}{\sqrt{\pi\mu(1-\mathrm{e}^{-2t/\sigma})\sigma^{-1}}} dx \leqslant \sqrt{\frac{\mu}{\pi\sigma}} < \infty.$$

Then usual theorems about martingale asymptotics prove that u_t converges almost surely to a random variable u_∞, and it is obvious that u_∞ must be a gaussian centered random variable with variance $\mu/2\sigma$. ◇

4.4. Asymptotic behaviour of $M_\delta(t)$

We are now able to prove the following assertion

Proposition 4.3.

$$\left\langle \mathbb{E}\left[T_0(\delta X_{\rho^{-2}t}, \delta Y_{\rho^{-2}t})\right]\right\rangle \to \mathrm{e}^{i\frac{1}{2}(\eta^2\mu+\xi^2(\nu+D(\varepsilon)))t}$$

where $D(\varepsilon)$ is a positive real number depending on ε, ψ_∞, μ and σ.

$\diamond$ As (20) shows, $\mathbb{E}[\langle Q_\delta(t)\rangle]$ has an upper bound independent on δ. then, we can assert that $M_\delta(t)$ converges in law to a martingale with quadratic variation $Q_0(t) = \lim Q_\delta(t)$. To compute this last limit, we use the ergodic theorem, noticing that u_t and e^{ikX_t} are ergodic. As

$$\mathbb{E}[\langle Q_\delta(t)\rangle] \to t\left[\mu\eta^2 + \frac{\xi^2\sigma^2}{2\pi\mu}\int_{\mathbb{R}} dk|k|^{1-\varepsilon}\psi_\infty(|k|)\mathbb{E}[|f_k'(u_\infty)|^2]\right]$$

$M_\delta(t)$ converges to a brownian motion with diffusion coefficient

$$\eta^2\mu + \xi^2 D(\varepsilon)$$

where

$$D(\varepsilon) = \frac{\sigma^2}{2\pi\mu}\int_{\mathbb{R}} dk|k|^{1-\varepsilon}\psi_\infty(|k|)\mathbb{E}[|f_k'(u_\infty)|^2].$$

Using (15), it is easy to show that

$$\begin{cases} \mathbb{E}\left[\left\langle |\xi\delta U_\delta(X_{\delta^{-2}t} + x/\delta, u_{\delta^{-2}t})|^2\right\rangle\right] \to 0 \\ \mathbb{E}\left[\left\langle |\xi\delta U_\delta(x/\delta, \delta u)|^2\right\rangle\right] \to 0 \end{cases}$$

when δ goes to 0, so that the limit of (19) is now known and the proposition is proved. $\diamond$

4.5. Computation of $D(\varepsilon)$

Let us compute $D(\varepsilon)$ by evaluating the limit of the variance of $\delta Y_{\delta^{-2}t}$.

Proposition 4.4.

$$D(\varepsilon)=\frac{1}{\pi}\int_{\mathbb{R}} dk|k|^{1-\varepsilon}\psi_\infty(|k|)\int_0^{+\infty} \mathrm{e}^{-\frac{k^2\mu\sigma}{2}(s/\sigma-1+\mathrm{e}^{-s/\sigma})}ds.$$

◇ Using (10), we manage to compute the variance of $\delta Y_{\delta^{-2}t}$

$$\begin{aligned}\left\langle V[\delta Y_{\delta^{-2}t}]\right\rangle &= \nu t-2\nu\tau\delta^2(1-\mathrm{e}^{-\delta^{-2}t/\tau})\\ &\quad +\frac{\delta^2\nu\tau}{2}(1-\mathrm{e}^{-2\delta^{-2}t/\tau})+I_\delta(\varepsilon,t)\end{aligned}\tag{21}$$

where

$$\begin{aligned}I_\delta(\varepsilon,t)=\frac{\delta^2}{2\pi}\int_{\mathbb{R}}\hat{F}_{\delta,\varepsilon}(k)dk\int_0^{\frac{t}{\delta^2}}\int_s^{2\frac{t}{\delta^2}-s}&\left(1-\mathrm{e}^{\left(\frac{s'+s}{2}-\frac{t}{\delta^2}\right)/\sigma}\right)\left(1-\mathrm{e}^{\left(\frac{s'-s}{2}-\frac{t}{\delta^2}\right)/\sigma}\right)\\ &\times\mathrm{e}^{-\frac{k^2\mu\sigma}{2}\left(s/\sigma-1+\mathrm{e}^{-s/\sigma}-(2-4\delta^2\frac{\sigma}{\mu}u^2)\mathrm{e}^{-s'/\sigma}\mathrm{sh}^2\frac{s}{2\sigma}\right)}ds'ds.\end{aligned}$$

Making an obvious change of variables, one obtains

$$\begin{aligned}&I_\delta(\varepsilon,t)=\\ &\frac{1}{2\pi}\int_{\mathbb{R}}\hat{F}_{\delta,\varepsilon}(k)dk\int_0^{\frac{t}{\delta^2}}\int_{\delta^2 s}^{2t-\delta^2 s}\left(1-\mathrm{e}^{\left(\frac{\delta^{-2}s'+s}{2}-\frac{t}{\delta^2}\right)/\sigma}\right)\\ &\qquad\times\left(1-\mathrm{e}^{\left(\frac{\delta^{-2}s'-s}{2}-\frac{t}{\delta^2}\right)/\sigma}\right)\\ &\qquad\times\mathrm{e}^{-\frac{k^2\mu\sigma}{2}\left(s/\sigma-1+\mathrm{e}^{-s/\sigma}-(2-4\delta^2\frac{\sigma}{\mu}u^2)\mathrm{e}^{-\delta^{-2}s'/\sigma}\mathrm{sh}^2\frac{s}{2\sigma}\right)}ds'ds.\end{aligned}$$

As

$$\begin{aligned}&\mathrm{e}^{-\frac{k^2\mu\sigma}{2}\left(s/\sigma-1+\mathrm{e}^{-s/\sigma}-(2-4\delta^2\frac{\sigma}{\mu}u^2)\mathrm{e}^{-\delta^{-2}s'/\sigma}\mathrm{sh}^2\frac{s}{2\sigma}\right)}\\ &\quad\leqslant \mathrm{e}^{-\frac{k^2\mu\sigma}{2}\left(s/\sigma-1+\mathrm{e}^{-s/\sigma}-2\mathrm{e}^{-\delta^{-2}s'/\sigma}\mathrm{sh}^2\frac{s}{2\sigma}\right)}\\ &\quad\leqslant \mathrm{e}^{-\frac{k^2\mu\sigma}{2}(s/\sigma-3/2+\mathrm{e}^{-s/\sigma})}\end{aligned}$$

for δ small enough, and for all $k \neq 0$

$$\int_0^{+\infty} \mathrm{e}^{-\frac{k^2\mu\sigma}{2}(s/\sigma-3/2+\mathrm{e}^{-s/\sigma})}ds \leqslant \frac{3\sigma}{2} + \int_{3\sigma/2}^{+\infty} \mathrm{e}^{-\frac{k^2\mu\sigma}{2}(s/\sigma-3/2)}ds$$

$$= \tfrac{3\sigma}{2} + \tfrac{2}{\mu k^2}$$

the Lebesgue's dominated convergence theorem applies and we have

$$\lim_{\delta\to 0} I_\delta(\varepsilon, t) = \frac{t}{\pi}\int_{\mathbb{R}} dk|k|^{1-\varepsilon}\psi_\infty(|k|)\int_0^{+\infty} \mathrm{e}^{-\frac{k^2\mu\sigma}{2}(s/\sigma-1+\mathrm{e}^{-s/\sigma})}ds. \ \diamondsuit$$

The proof of Theorem 1 is now complete.

5. THE SUPERCONVECTIVE $\varepsilon>2$ CASE

Let us go back to (11). Again, previous works by M. Avellaneda and A. Majda[1] let us suppose that the behaviour of the rescaled process will be superconvective. Let us make the hypothesis

$$\frac{\delta}{\rho} \to 0. \tag{22}$$

Under this hypothesis, easy calculus show that

$$\begin{cases} \delta X_{\rho^{-2}t} \to 0 \\ \rho^2(1-\mathrm{e}^{-\rho^{-2}t/\tau})v \to 0 \\ \delta\displaystyle\int_0^{\rho^{-2}t}(1-\mathrm{e}^{(s-\rho^{-2}t)/\tau})d\beta'_s \to 0 \\ \delta\displaystyle\int_0^{\rho^{-2}t}\mathrm{e}^{(s-\rho^{-2}t)/\tau}\gamma_\delta(X_s+\delta^{-1}x)ds \to 0 \end{cases} \tag{23}$$

all of these convergences holding in L^2 and pointwise in x, y, u, v. Considering this, the only remaining term we have to deal with is

$$\left\langle \mathbb{E}\left[\mathrm{e}^{i\xi\delta\int_0^{\rho^{-2}t}\gamma_\delta(X_s+\delta^{-1}x)ds}\right]\right\rangle.$$

Using Fubini's theorem and the fact that γ_δ is gaussian, we obtain

$$\left\langle \mathbb{E}\left[e^{i\xi\delta\int_0^{\rho^{-2}t}\gamma_\delta(X_s+\delta^{-1}x)ds}\right]\right\rangle = \mathbb{E}\left[e^{-\frac{1}{2}\xi^2\delta^2\left\langle\left|\int_0^{\rho^{-2}t}\gamma_\delta(X_s+\delta^{-1}x)ds\right|^2\right\rangle}\right]. \quad (24)$$

As (3) gives the statistics of γ_δ, we obtain the following equation

$$\delta^2\left\langle\left|\int_0^{\rho^{-2}t}\gamma_\delta(X_s+\delta^{-1}x)ds\right|^2\right\rangle = \frac{\delta^2}{2\pi}\int_0^{\rho^{-2}t}\int_0^{\rho^{-2}t} ds ds' \int_{\mathbb{R}} e^{ik(X_s-X_{s'})}\hat{F}_{\delta,\varepsilon}(k)dk$$

where $\hat{F}_{\delta,\varepsilon}(k)$ is defined by (4). Let us make the change of variables $s := \rho^2 s$ and $s' := \rho^2 s'$ in the double integral, and then $k := k/\delta$ in the other one. We obtain the following equality

$$\begin{aligned}&\frac{\delta^2}{2\pi}\int_0^{\rho^{-2}t}\int_0^{\rho^{-2}t} ds ds' \int_{\mathbb{R}} e^{ik(X_s-X_s')}\hat{F}_{\delta,\varepsilon}(k)dk \\ &= \frac{\delta^{4-\varepsilon}}{2\pi\rho^4}\int_{\mathbb{R}}|k|^{1-\varepsilon}\psi_0(|k|)\psi_\infty(\delta|k|)\int_0^t\int_0^t ds ds' e^{ik\delta(X_{s/\rho^2}-X_{s'/\rho^2})}dk.\end{aligned}$$

Using the fact that $\delta X_{\rho^{-2}t}\to 0$ almost surely, the right scaling factor is

$$\rho = \delta^{1-\varepsilon/4}. \quad (25)$$

To ensure that $\rho\to 0$, we see that we need to restrain our study to the case

$$2<\varepsilon<4.$$

Then the dominated convergence theorem applies and we obtain

$$\left\langle \mathbb{E}\left[e^{i\xi\delta\int_0^{\rho^{-2}t}\gamma_\delta(X_s+\delta^{-1}x)ds}\right]\right\rangle \to e^{-\frac{1}{4\pi}\xi^2t^2\int_{\mathbb{R}}|k|^{1-\varepsilon}\psi_0(|k|)dk}.$$

Notice that (25) is coherent with the hypothesis (22). The proof of Theorem 2 is done.

6. THE ANOMALOUS REGIME IN THE 0<ε<2 CASE

6.1. The Scale Factor

For the same reason as in the previous case, we suppose that

$$\frac{\delta}{\rho} \to 0. \tag{26}$$

and then, (23) shows again that we only have to study the term

$$\left\langle \mathbb{E}\left[\mathrm{e}^{i\xi\delta\int_0^{\rho^{-2}t}\gamma_\delta(X_s+\delta^{-1}x)ds} \right] \right\rangle$$

which, using (24), is still equal to

$$\mathbb{E}\left[\mathrm{e}^{-\frac{1}{2}\xi^2\frac{\delta^2}{2\pi}\int_0^{\rho^{-2}t}\int_0^{\rho^{-2}t}dsds'\int_{\mathbb{R}}\mathrm{e}^{ik(X_s-X_{s'})}\hat{F}_{\delta,\varepsilon}(k)dk} \right]. \tag{27}$$

Let us make the change of variables $s := \rho^2 s/t$, $s' := \rho^2 s'/t$, $k := k\sqrt{t}/\rho$. We obtain

$$\frac{\delta^2}{2\pi}\int_0^{\rho^{-2}t}\int_0^{\rho^{-2}t} dsds' \int_{\mathbb{R}} \mathrm{e}^{ik(X_s-X_{s'})}|k|^{1-\varepsilon}\psi_0\left(\frac{|k|}{\delta}\right)\psi_\infty(|k|)dk$$
$$= \frac{t^{1+\frac{\varepsilon}{2}}\delta^2}{2\pi\rho^{2+\varepsilon}}\int_0^1\int_0^1 dsds' \int_{\mathbb{R}} \mathrm{e}^{i\frac{k\rho}{\sqrt{t}}(X_{st/\rho^2}-X_{s't/\rho^2})}|k|^{1-\varepsilon}\psi_0\left(\frac{\rho|k|}{\delta\sqrt{t}}\right)\psi_\infty\left(\frac{\rho|k|}{\sqrt{t}}\right)dk$$

This incites us to choose the following scaling ρ

$$\rho = \delta^{\frac{1}{1+\varepsilon/2}}$$

which is coherent with (26). As $X_t = \sqrt{\mu}\beta_t - \sigma u_t + \delta\sigma u$, we have, making again the change of variable $k := k\sqrt{\mu}$

$$-\frac{\xi^2 t^{1+\frac{\varepsilon}{2}}}{4\pi}\int_0^1\int_0^1 dsds' \int_{\mathbb{R}} \mathrm{e}^{i\frac{k\rho}{\sqrt{t}}(X_{st/\rho^2}-X_{s't/\rho^2})}|k|^{1-\varepsilon}\psi_0\left(\frac{\rho|k|}{\delta\sqrt{t}}\right)\psi_\infty\left(\frac{\rho|k|}{\sqrt{t}}\right)dk$$
$$= -\frac{\xi^2\mu^{\frac{\varepsilon}{2}-1}t^{1+\frac{\varepsilon}{2}}}{4\pi}\int_0^1\int_0^1 dsds'\int_{\mathbb{R}}\mathrm{e}^{i\frac{k\rho}{\sqrt{t}}(\beta_{st/\rho^2}-\beta_{s't/\rho^2})}\hat{F}'_{\delta,\varepsilon}(k)dk$$

where

$$\hat{F}'_{\delta,\varepsilon}(k) = \mathrm{e}^{i\frac{k\rho\sigma}{\sqrt{\mu t}}(u_{st/\rho^2}-u_{s't/\rho^2})}|k|^{1-\varepsilon}\psi_0\left(\frac{\rho|k|}{\delta\sqrt{\mu t}}\right)\psi_\infty\left(\frac{\rho|k|}{\sqrt{\mu t}}\right).$$

6.2. Introduction of a Suitable Measure

Let B_s be the standard Wiener process equal to the almost sure limit of $\frac{\rho}{\sqrt{t}}\beta_{st/\rho^2}$ when δ goes to 0. Let us define on $\mathbb{R}$ the Borel measure μ_B via the formula

$$\int_{\mathbb{R}} g(x)d\mu_B(x) = \int_0^1 \int_0^1 g(B_s - B_{s'})dsds'.$$

One can see that the function

$$\hat{G}_\delta(k) = |k|^{1-\varepsilon}\, 1\!\!1_{\delta<|k|<\delta^{-1}}$$

belongs to every L^p for $1 \leqslant p \leqslant \infty$, so it has an inverse Fourier transform $G_\delta(x)$ which verifies, in the sense of tempered distributions, when $\delta \to 0$

$$G_\delta(x) \to G(x) \begin{cases} C(\varepsilon)|x|^{\varepsilon-2} & \text{for} \quad \varepsilon \neq 1 \\ C(1)\delta(x) & \text{for} \quad \varepsilon = 1 \end{cases}$$

where $C(\varepsilon)$ is defined by

$$\begin{cases} C(\varepsilon) = 2(2\pi)^{-1/2} \sin\left(\frac{(1-\varepsilon)}{2}\right) \Gamma(2-\varepsilon) & \text{for} \quad \varepsilon \neq 1 \\ C(1) = (2\pi)^{-1/2} \end{cases} \tag{28}$$

In order to achieve the proof of Theorem 3, we will need the two following results

Lemma 6.1. Let ϕ be in the Sobolev space $H^{3/2-\lambda}(\mathbb{R})$ for all $\lambda > 0$, with its Fourier transform continuous and bounded. Then

$$\lim_{\delta\to 0} \int_{\mathbb{R}} \phi(x) G_\delta(x) dx = [G, \phi] \tag{29}$$

where [., .] denotes the action of a tempered distribution and $H^s(\mathbb{R})$ consists in functions ϕ so that $|\hat{\phi}|(1+|k|^2)^{s/2} \in \mathrm{L}^2(\mathbb{R})$. Furthermore, there exist λ and K depending only on ε so that

$$|[G_\delta, \phi]| \leqslant K(\|\phi\|_{H^{3/2-\lambda}} + \|\hat{\phi}\|_\infty).$$

Proposition 6.2. The following assertions hold

- The measure μ_B is absolutely continuous with respect to Lebesgue measure with density ϕ_B
- This density is in the Sobolev space $H^{3/2-\lambda}(\mathbb{R})$ for all $\lambda > 0$ (in particular, it is α-Hölder for any $\alpha < 1$) and its Fourier transform is continuous and bounded.

$\diamond$ The proofs of these technical results are contained in ref. 1, pp. 400–404. $\diamond$

Then, using these facts and dominated convergence theorem, we have when δ goes to 0

$$\int_0^1\int_0^1 ds ds' \int_{\mathbb{R}} e^{i\frac{k\rho}{\sqrt{t}}(\beta_{st/\rho^2}-\beta_{s't/\rho^2})} \hat{F}'_{\delta,\varepsilon}(k) \mathbb{1}_{\delta<|k|<\delta^{-1}} dk \rightarrow \int_{\mathbb{R}} \hat{\mu}_B(k)\hat{G}(k)dk \quad (30)$$

Using these properties and Plancherel's theorem on (30), one can easily show that the exponential inside the expactation in (27) converges to the following quantities

- For $0 < \varepsilon < 1$

$$e^{-\frac{\xi^2}{\sqrt{8\pi}} t^{1+\varepsilon/2} \mu^{\varepsilon/2-1} C(\varepsilon) \int_0^1 \int_0^1 |B_s - B_{s'}|^{\varepsilon-2} ds ds'}$$

- For $1 < \varepsilon < 2$

$$e^{-\frac{\xi^2}{\sqrt{8\pi}} t^{1+\varepsilon/2} \mu^{\varepsilon/2-1} C(\varepsilon) \int_{\mathbb{R}} \frac{\phi_B(x) - \phi_B(0)}{|x|^{2-\varepsilon}} dx}$$

  The convergence of this integral being ensured by the fact that ϕ_B is $\alpha-$Hölder for $\alpha < 1$.
- For $\varepsilon = 1$

$$e^{-\frac{\xi^2}{\sqrt{8\pi}} t^{3/2} \mu^{-1/2} \phi_B(0)}$$

6.3. The Diffusion Equation

Going back to (11), we have proved the following formula

$$\bar{T}(x,y,t) = \frac{1}{\sqrt{2\pi}} \iint_{\mathbb{R}^2} e^{i(\eta x + \xi y)} \hat{T}_0(\eta,\xi) \mathbb{E}\left[e^{-\frac{\xi^2}{\sqrt{8\pi}} t^{1+\varepsilon/2} \mu^{\varepsilon/2-1} [G,\phi_B]} \right] d\eta d\xi$$

We introduce the measure $d\nu_\varepsilon(\alpha)$, which we can call the *random diffusivity*, as the distribution of the random variable $[G, \phi_B]$

$$\nu_\varepsilon(\alpha) = \mathbb{E}\left[[G,\phi_B] \leqslant \alpha \right]$$

Still refering to ref. 1, it is easy to show that $\nu_\varepsilon(\alpha)$ vanishes for $\alpha \leqslant 0$, and then $\bar{T}$ verifies

$$\bar{T}(x,y,t) = \int_0^\infty T_\alpha(x,y,t) \nu_\varepsilon(d\alpha)$$

where, for fixed $\alpha > 0$, T_α is the solution of

$$\partial_t T_\alpha = \left(1 + \frac{\varepsilon}{2}\right) \frac{t^{\varepsilon/2} \mu^{\varepsilon/2-1}}{\sqrt{8\pi}} \alpha \partial^2_{y^2} T_\alpha.$$

Proof of Theorem 3 is therefore complete.

6.4. Computation of The Effective Variance

In order to have an idea of the speed of the diffusion in this case, let us compute

$$\lim_{\delta \to 0} \left\langle V[\delta Y_{\rho^{-2}t}] \right\rangle.$$

Refering to (21), we obtain

$$\left\langle V[\delta Y_{\rho^{-2}t}] \right\rangle = \frac{\delta^2 \nu}{\rho^2} t - 2\nu\tau\delta^2 (1 - e^{-\rho^{-2}t/\tau}) + \frac{\nu\tau\delta^2}{2} (1 - e^{-2\rho^{-2}t/\tau}) + I_\delta(\varepsilon, t)$$

where

$$I_\delta(\varepsilon,t)=\frac{\delta^2}{2\pi}\int_{\mathbb{R}}\hat{F}_{\delta,\varepsilon}(k)dk\int_0^{\frac{t}{\rho^2}}\int_s^{2\frac{t}{\rho^2}-s}\left(1-\mathrm{e}^{\left(\frac{s'+s}{2}-\frac{t}{\rho^2}\right)/\sigma}\right)\left(1-\mathrm{e}^{\left(\frac{s'-s}{2}-\frac{t}{\rho^2}\right)/\sigma}\right)$$

$$\times\mathrm{e}^{-\frac{k^2\mu\sigma}{2}\left(s/\sigma-1+\mathrm{e}^{-s/\sigma}-(2-4\delta^2\frac{\sigma}{\mu}u^2)\mathrm{e}^{-s'/\sigma}\mathrm{sh}^2\frac{s}{2\sigma}\right)}ds'ds.$$

Now, simply by making $k:=k\sqrt{t\mu}/\rho$, $s:=\rho^2 s$, $s':=\rho^2 s'$, and applying the dominated convergence theorem, one obtains

$$\lim_{\delta\to 0}\left\langle V[\delta Y_{\rho^{-2}t}]\right\rangle=\left(\frac{2\mu^{\varepsilon/2-1}}{\pi}\int_{\mathbb{R}}|k|^{1-\varepsilon}\left(\frac{1}{k^2}-\frac{2}{k^4}(1-\mathrm{e}^{-k^2/2})\right)dk\right)t^{1+\frac{\varepsilon}{2}}$$

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